Stochastic Ordering based Carrier-to-Interference Ratio Analysis for the Shotgun Cellular Systems

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Abstract

A simple analytical tool based on stochastic ordering is developed to compare the distributions of carrier-to-interference ratio at the mobile station of two cellular systems where the base stations are distributed randomly according to certain non-homogeneous Poisson point processes. The comparison is conveniently done by studying only the base station densities without having to solve for the distributions of the carrier-to-interference ratio, that are often hard to obtain.

Index Terms: Carrier-to-interference Ratio, Co-channel Interference, Fading channels, Stochastic ordering.

I. Introduction

A Poisson point process has been adopted in the literature for the locations of nodes in the study of shotgun cellular networks, ad-hoc networks, and other uncoordinated and decentralized communication networks [1], [2, and references therein]. An underlying assumption in all the previous work is that the density of transmitters, referred to as base station (BS) throughout this paper, is constant, i.e. the Poisson point process is homogeneous. Such a model does not sufficiently represent reality.

In [2] and here, we have modeled BS arrangement by non-homogeneous Poisson point processes in $\mathbb{R}^l,\ l = 1, 2,\ \text{and}\ 3$. We aim to characterize carrier-to-interference ratio $\left(\frac{C}{I}\right)$ at a given

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mobile station (MS). In [2], we have derived semi-analytical expressions for the tail probability of $\frac{C}{I}$, denoted as $\mathbb{P}\left(\left\{\frac{C}{I} > y\right\}\right)$, by deriving the characteristic function of the reciprocal of $\frac{C}{I}$. Moreover, the $\frac{C}{I}$ characterization holds for a wide range of scenarios of interest such as arbitrary distributions for fading and random BS transmission powers, arbitrary path-loss models and arbitrary locations for the MS. In spite of such a general result, it is still not convenient to compare the $\frac{C}{I}$ distributions of two different networks.

Is it possible to qualitatively compare two $\frac{C}{I}$ distributions by only examining the BS densities without having to obtain the $\frac{C}{I}$ distributions? This paper answers the question affirmatively for certain BS densities by developing a stochastic ordering based tool. Concepts of stochastic ordering have been applied to scenarios of interest in wireless communications in [3]. To the best of our knowledge, this is the first work that uses stochastic ordering to understand large scale random wireless networks. The main result of this paper is Theorem 1 for which Section III develops the necessary tools. The utility of this result is explored in Section IV, by considering several scenarios of interest in modeling the wireless network.

II. System Model

The *Shotgun Cellular System (SCS)* is a model for the cellular system in which the BSs are distributed in a l-dimensional plane ($l-$D, typically $l = 1, 2, \text{ and } 3$) according to a non-homogeneous Poisson point process in $\mathbb{R}^l$. The intensity function of the Poisson point process is called the BS density function.

Without loss of generality, we restrict our attention to 1-D SCSs, because for the $\frac{C}{I}$ analysis, the l-D SCSs can be reduced to an equivalent 1-D SCS [2, Lemma 2] with a BS density function $\lambda(r)$, where $r \geq 0$ is the distance of the BS from a mobile-station (MS) located at the origin. For example, a *homogeneous* l-D SCS with density $\lambda_0\,(> 0)$ is equivalent to a 1-D SCS with density function $\lambda\,(r) = \lambda_0 b_l r^{l-1},\ r \geq 0,\ b_1 = 2,\ b_2 = 2\pi,\ \text{and}\ b_3 = 4\pi$ [2, Corollary 1].

The BSs are assumed to have independent and identically distributed (i.i.d.) random transmission powers K_i's and shadow fadings Ψ_i's across BSs. The deterministic path-loss is $R^{-\varepsilon},\ \varepsilon > 0$. We assume an interference limited system and omit background noise. We focus on the signal quality of a MS at the origin. The MS chooses to communicate with the BS that corresponds to the strongest received signal power, referred to as the "serving BS". All other BSs are the "interfering BSs". The signal quality at the MS is measured by $\frac{C}{I} = \frac{K_S \Psi_S R_S^{-\varepsilon}}{\sum_{i=1}^{\infty} K_i \Psi_i R_i^{-\varepsilon}}$, where S

indexes the serving BS and i indexes the interfering BSs. Further, $R_S \leq R_1 \leq R_2 \leq \cdots$ are ordered BS locations.

III. The Stochastic Ordering of $\frac{C}{I}$

In this section, we present the theoretical tools that are used to compare $\frac{C}{I}$ tail probability by comparing the equivalent 1-D BS densities $\lambda(r)$. Since the effect of i.i.d. shadow fading factors and i.i.d. transmission powers can be captured by modifying the BS density as shown in Section IV-D, they are assumed to be 1 for all BSs. The generalization to arbitrary path loss model is given in [2, Section VI], which is also equivalent to modifying $\lambda(r)$. As a result, $\frac{C}{I} = \frac{R_1^{-\varepsilon}}{\sum_{i=2}^{\infty} R_i^{-\varepsilon}}$.

Definition 1. Let X and Y be two random variables such that $\mathbb{P}\left(\{X > x\}\right) \leq \mathbb{P}\left(\{Y > x\}\right),\ \forall\, x \in (-\infty, \infty)$, then X is *smaller than* Y *in the usual stochastic order* and this is denoted by $X \leq_{\text{st}} Y$. Further, $X =_{\text{st}} Y$ means $\mathbb{P}\left(\{X > x\}\right) = \mathbb{P}\left(\{Y > x\}\right),\ \forall\, x \in (-\infty, \infty)$. [4, p. 3]

If X and Y are the $\frac{C}{I}$ at the MS in two different SCSs, $X \leq_{\text{st}} Y$ implies that the MS in the SCS corresponding to Y is more likely to achieve better signal quality than in the SCS corresponding to X.

Let $\{R_k\}_{k=1}^{\infty}$ represent the set of distances of BSs from the MS (indexed in the ascending order of the distance), $D_{k+1} = R_{k+1} - R_k$ be the distance between two adjacent BSs, and $f_{D_{k+1}|R_k}\left(d|\, r; \lambda(s)\right)$ be the probability density function (p.d.f.) of D_{k+1} conditioned on $R_k = r$, as a function of the BS density $\lambda(s)$.

Lemma 1.

$$\begin{aligned} f_{D_{k+1}|R_k}\left(d\,|r; \lambda(s)\right) &= e^{-\int_r^{r+d} \lambda(s) ds} \lambda(r+d), \text{ and} && (1)\\ f_{aD_{k+1}|aR_k}\left(d'\,|r'; \lambda(s)\right) &= f_{D_{k+1}|R_k}\left(d' \middle| r'; \frac{1}{a}\lambda\left(\frac{s}{a}\right)\right). && (2)\end{aligned}$$

Proof: Equation (1) follows from the properties of Poisson processes [5], [6]. Equation (**2**) is proved by $f_{aD_{k+1}|aR_k}\left(d'|r'; \lambda(s)\right) \overset{(a)}{=} \frac{1}{a} f_{D_{k+1}|R_k}\left(\frac{d'}{a}|\frac{r'}{a}; \lambda(s)\right) \overset{(b)}{=} \frac{1}{a}\lambda\left(\frac{r'+d'}{a}\right) \exp\left(-\int_{\frac{r'}{a}}^{\frac{r'+d'}{a}} \lambda(s) ds\right)$ $\overset{(c)}{=} \frac{1}{a}\lambda\left(\frac{r'+d'}{a}\right) \exp\left(-\int_{r'}^{r'+d'} \frac{1}{a}\lambda\left(\frac{s'}{a}\right) ds'\right)$, where (a) is obtained by a variable change; (b) follows from (1); and (c) is obtained by a variable change and gives (2). ■

Lemma 1 means that scaling D_{k+1} and R_k by a is equivalent to scaling the BS density as $\frac{1}{a}\lambda(\frac{r}{a})$. The significance of Lemma 1 in the context of $\frac{C}{I}$ is as follows.

Corollary 1. The distribution of $\frac{C}{I}$ at the MS in the 1-D SCS with BS density function $\lambda(r)$ is the same as that in 1-D SCSs with BS density functions $\frac{1}{a}\lambda(\frac{r}{a})$, $\forall\, a > 0$, i.e., $\frac{C}{I}\big|_{\lambda(r)} =_{\text{st}} \frac{C}{I}\big|_{\frac{1}{a}\lambda(\frac{r}{a})}$.

Proof: Let $\{R_k\}_{k=1}^{\infty}$ correspond to the 1-D SCS with BS density function $\lambda(r)$. Then, since the ordered BS locations R_k's are determined by inter-BS distances, it follows from Lemma 1 that $\frac{C}{I}\big|_{\lambda(r)} = \frac{(aR_1)^{-\varepsilon}}{\sum_{k=2}^{\infty}(aR_k)^{-\varepsilon}}\Big|_{\lambda(r)} =_{\text{st}} \frac{\left(R_1'\right)^{-\varepsilon}}{\sum_{k=2}^{\infty}\left(R_k'\right)^{-\varepsilon}}\Bigg|_{\frac{1}{a}\lambda(\frac{r}{a})}$, where R_k''s corresponding to $\frac{1}{a}\lambda(\frac{r}{a})$ have the same distribution as aR_k's with $\lambda(r)$. ∎

As a result, $\left\{\frac{1}{a}\lambda\left(\frac{r}{a}\right),\ r \geq 0,\ a > 0\right\}$ forms a parametric family of BS density functions such that the 1-D SCSs corresponding to them have the same $\frac{C}{I}$ at the MS. In other words, appropriately scaling the BS density function will not change the p.d.f. of $\frac{C}{I}$. Moreover, the following special case is a direct corollary of the above result.

Corollary 2. In a *homogeneous* l-D SCS, $\frac{C}{I}$ is not a function of the BS density.

Proof: Firstly, recall that the $\frac{C}{I}$ at the MS in a *homogeneous* $l-$D SCS with BS density λ_0 is the same as that in a $1-$D SCS with a BS density function $\lambda\left(r\right) = \lambda_0 b_l r^{l-1}$. Next, from Corollary 1, the distribution of $\frac{C}{I}$ in this SCS is the same as that in a 1-D SCS with the BS density function $\frac{1}{a}\lambda\left(\frac{r}{a}\right) = \lambda_0 \alpha b_l r^{l-1},\ \alpha = a^{-l},\ a > 0$. Thus, distributions of $\frac{C}{I}$ corresponding to $\alpha\lambda_0$ and λ_0 are the same. ∎

This was also observed in [2], where we showed that the expression for the characteristic function of $\left(\frac{C}{I}\right)^{-1}$ did not involve λ_0. Corollary 2 provides a simpler and more fundamental proof. Next, we define a notation used in Theorem 1.

Definition 2. For BS density function $\lambda\left(r\right)$, the cumulative BS density function is defined as $\mu(r) \triangleq \int_0^r \lambda(s)ds$, and its inverse function is define as $\mu^{-1}(q) \triangleq \sup\{r : \mu(r) \leq q\}$.

Since $\lambda(r) \geq 0$, $\mu\left(r\right)$ is a monotonically increasing function of r. In general, the inverse function is not injective since $\lambda(r)$ can be zero in arbitrary intervals of $r \in [0, \infty)$. The above definition makes it injective. For certain BS densities, it is possible to compare two $\frac{C}{I}$'s by comparing the densities without solving for the distributions. It is facilitated by Theorem 1.

Theorem 1. Let $\left\{\lambda_1\left(r\right), \mu_1\left(r\right), \mu_1^{-1}\left(q\right)\right\}$ and $\left\{\lambda_2\left(r\right), \mu_2\left(r\right), \mu_2^{-1}\left(q\right)\right\}$ be the BS density functions, cumulative BS density functions and their inverse functions for two 1-D SCSs, respectively.

The $\frac{C}{I}$ at the MS follows the stochastic order $\frac{C}{I}\big|_{\lambda_1(r)} \leq_{\text{st}} \frac{C}{I}\big|_{\lambda_2(r)}$, if for each $q > 0$ and $a = \frac{\mu_2^{-1}(q)}{\mu_1^{-1}(q)}$, $\frac{1}{a}\lambda_1\left(\frac{r}{a}\right) \geq \lambda_2(r),\ \forall\ r \geq \mu_2^{-1}(q)$.

See Appendix A for the proof. Applications of the above theorem are in the next section.

IV. Applications of the $\frac{C}{I}$ stochastic ordering

A. Comparison of Homogeneous l–D SCSs $(l = 1, 2,\ \text{and}\ 3)$

Here, we show that the signal quality degrades as the dimension l of the *homogeneous* l-D SCS increases, for which we need the following corollaries.

Corollary 3. For each $q > 0$ and $a = \frac{\mu_2^{-1}(q)}{\mu_1^{-1}(q)}$, if $\Delta\lambda(r) \triangleq \frac{1}{a}\lambda_1(\frac{r}{a}) - \lambda_2(r)$ is a non-decreasing function for all $r \geq 0$, then $\frac{C}{I}\big|_{\lambda_1(r)} \leq_{\text{st}} \frac{C}{I}\big|_{\lambda_2(r)}$.

Proof: Note that $\int\limits_0^{\mu_2^{-1}(q)} \frac{1}{a}\lambda_1\left(\frac{s}{a}\right) ds = q = \int\limits_0^{\mu_2^{-1}(q)} \lambda_2(s) ds$. Hence, $\int\limits_0^{\mu_2^{-1}(q)} \Delta\lambda(s) ds = 0$. Suppose $\Delta\lambda(\mu_2^{-1}(q)) < 0$, then $\Delta\lambda(r) < 0,\ r \in [0, \mu_2^{-1}(q)]$, since $\Delta\lambda(r)$ is non-decreasing. This is a contradiction. Thus, $\Delta\lambda(\mu_2^{-1}(q)) \geq 0$. Using Theorem 1, the corollary is proved. ■

Corollary 4. For a *homogeneous* l-D SCS with BS density λ_0 and its equivalent 1-D BS density function $\lambda_l(r) = \lambda_0 b_l r^{l-1},\ r \geq 0$, multiplying $\lambda_l(r)$ with a non-increasing function $\beta(r) > 0$ improves the $\frac{C}{I}$, i.e., $\frac{C}{I}\big|_{\lambda_l(r)} \leq_{\text{st}} \frac{C}{I}\big|_{\beta(r)\lambda_l(r)}$. The inequality reverses if $\beta(r)$ is non-decreasing.

Proof: If $\beta(r)$ is non-increasing, for any $a > 0$, the density difference $\Delta\lambda(r) = \frac{1}{a}\lambda_l(\frac{r}{a}) - \beta(r)\lambda_l(r) = \lambda_0 b_l \left(\frac{1}{a^{l-2}} - \beta(r)\right) r^{l-1}$ is non-decreasing. By Corollary 3, $\frac{C}{I}\big|_{\lambda_l(r)} \leq_{\text{st}} \frac{C}{I}\big|_{\beta(r)\lambda_l(r)}$ holds. If $\beta(r)$ is non-decreasing, the same proof applies with $\Delta\lambda(r) = \beta(r)\lambda_l(r) - a\lambda_l(\frac{r}{a})$. ■

Hence, $\frac{C}{I}\big|_{\lambda_1(r)} \overset{(a)}{\geq_{\text{st}}} \frac{C}{I}\big|_{\lambda_2(r)} \overset{(b)}{\geq_{\text{st}}} \frac{C}{I}\big|_{\lambda_3(r)}$ by plugging $l = 1, 2, 3$ in $\lambda_l(r)$, respectively; (a) holds because $\lambda_2(r) = \beta(r)\lambda_1(r)$, where $\beta(r) = \frac{b_2}{b_1} r$ is a non-decreasing function; and similarly (b) also holds. Thus, the comparison between $\frac{C}{I}' s$ is done without finding their distributions.

B. A Qualitative Comparison between Two 1-D SCS

Consider a *homogeneous* 1-D SCS with a BS density function $\lambda_1(r) = \lambda,\ r \geq 0$, and another 1-D SCS with a BS density function $\lambda_2(r) = \begin{cases} \alpha & 0 \leq r \leq \rho \\ \beta & r > \rho \end{cases}$, where $\alpha > \beta$. Such $\lambda_2(r)$ might describe, for example, a highway passing through a region of greater population

(BS density of α) and then a region of smaller population (BS density of β). Usually, such a scenario is approximated by a constant BS density through out the highway, which is represented by $\lambda_1(r)$. If $\lambda = \alpha$, it is easy to guess that $\frac{C}{I}\big|_{\lambda_1(r)} \leq_{\text{st}} \frac{C}{I}\big|_{\lambda_2(r)}$. But if $\lambda > \alpha$, it is not clear which SCS has better $\frac{C}{I}$. Theorem 1 shows that the SCS with density $\lambda_2(r)$ has better $\frac{C}{I}$ than $\lambda_1(r)$, and the result holds irrespective of the specific values of $\alpha,\ \beta$, and λ. To apply Theorem 1, we note that $\mu_1^{-1}(q) = \frac{q}{\lambda},\ q \geq 0,$ and $\mu_2^{-1}(q) = \begin{cases} \frac{q}{\alpha} & ,\ q \leq \alpha\rho \\ \frac{q+(\beta-\alpha)\rho}{\beta} & ,\ q > \alpha\rho \end{cases}$. Further, $a(q) = \begin{cases} \frac{\lambda}{\alpha} & ,\ q \leq \alpha\rho, \\ \frac{1+(\beta-\alpha)\rho/q}{\beta/\lambda} & ,\ q > \alpha\rho \end{cases}$. As a result, for $q \leq \alpha\rho$, $\frac{1}{a}\lambda_1(\frac{r}{a}) = \begin{cases} \alpha \geq \alpha = \lambda_2(r) & ,\ \frac{q}{\alpha} < r < \rho \\ \alpha \geq \beta = \lambda_2(r) & ,\ r > \rho \end{cases}$, and for $q > \alpha\rho$, $\mu_2^{-1}(q) > \rho$, $\frac{1}{a}\lambda_1(\frac{r}{a}) = \frac{\beta}{1+(\beta-\alpha)\rho/q} \geq \beta = \lambda_2(r),\ r > \rho$. Thus, applying Theorem 1, $\frac{C}{I}\big|_{\lambda_1(r)} \leq_{\text{st}} \frac{C}{I}\big|_{\lambda_2(r)}$. Similarly, if $\alpha < \beta$, we can show $\frac{C}{I}\big|_{\lambda_1(r)} \geq_{\text{st}} \frac{C}{I}\big|_{\lambda_2(r)}$.

C. Comparison of Path-loss Models

Here, we compare the $\frac{C}{I}$ at the MS in two *homogeneous* $l-$D SCSs with a BS density λ_0 and with different path-loss models, $\frac{1}{h_1(r)}$, and $\frac{1}{h_2(r)}$, $\forall\ r \geq 0$. The proofs for Corollary 5 and 6 involve the following common steps. Firstly, using [2, Lemma 2], reduce the homogeneous $l-$D SCS to the equivalent 1-D SCS with a BS density function $\lambda_i(r) = \lambda_0 b_l r^{l-1},\ i = 1,\ 2.$ Next, using [2, Theorem 5], the resultant 1-D SCSs can be reduced to an equivalent 1-D SCS with path-loss exponent $\varepsilon = 1$, i.e., $\frac{1}{h_i(r)} = \frac{1}{r}$, and with BS density functions $\bar{\lambda}_i(r) = \frac{\lambda_i\left(h_i^{-1}(r)\right)}{h_i'\left(h_i^{-1}(r)\right)}$, $i = 1, 2$ where $h_i^{-1}(\cdot)$ is the inverse function of $h_i(\cdot)$. Finally, we have two 1-D SCSs with the same path-loss model $\frac{1}{h_i(r)} = \frac{1}{r}$ and BS density functions $\bar{\lambda}_i(r),\ r \geq 0,\ i = 1, 2$. We show that Theorem 1 applies and derive the result.

Corollary 5. In a *homogeneous* $l-$D SCS, if the path-loss follows a power-law parametrized by a path-loss exponent, ε, the $\frac{C}{I}$ at the MS improves as the path-loss exponent increases. In other words, if $h_i(r) = r^{\varepsilon_i},\ i = 1, 2,$ such that $\varepsilon_1 > \varepsilon_2 > l$, then $\left(\frac{C}{I}\right)_1 \geq_{\text{st}} \left(\frac{C}{I}\right)_2$, where $\left(\frac{C}{I}\right)_i$ corresponds to the path-loss model $\frac{1}{h_i(r)}$.

Proof: At the end of Step 2, the equivalent 1-D SCSs with a path-loss model $\frac{1}{r}$ have the BS density functions $\bar{\lambda}_i(r) = \frac{\lambda_0 b_l}{\varepsilon_i} r^{\frac{l}{\varepsilon_i}-1}$. Further, $\bar{\lambda}_2(r) = \beta(r)\bar{\lambda}_1(r)$, where $\beta(r) = \frac{\varepsilon_1}{\varepsilon_2} r^{\frac{l}{\varepsilon_2}-\frac{l}{\varepsilon_1}},\ r \geq 0$ is a non-decreasing function. Hence, Corollary 4 applies and $\frac{C}{I}\big|_{\bar{\lambda}_1(r)} \geq_{\text{st}} \frac{C}{I}\big|_{\bar{\lambda}_2(r)}$. ■

Hence, a simple proof that does not require solving the distribution of $\frac{C}{I}$ gives the expected result that a channel with a greater path-loss exponent has a better $\frac{C}{I}$. The following corollary establishes a similar result between two popularly used path-loss models [7].

Corollary 6. In a *homogeneous* $l-$D SCS with a BS density λ_0, the received signal of a MS located at the origin satisfies $\left(\frac{C}{I}\right)_1 \leq_{\text{st}} \left(\frac{C}{I}\right)_2$, where $\left(\frac{C}{I}\right)_1$ corresponds to path-loss $\frac{1}{h_1(r)}$ with $h_1(r) = r^{\varepsilon_1},\ r \geq 0$ and $\left(\frac{C}{I}\right)_2$ corresponds to the path-loss $\frac{1}{h_2(r)}$ with $h_2(r) = \begin{cases} r^{\varepsilon_1} & ,\ r \leq 1 \\ r^{\varepsilon_2} & ,\ r > 1 \end{cases}$, where $\varepsilon_2 > \varepsilon_1 > l$. The opposite conclusion holds when $\varepsilon_1 > \varepsilon_2 > l$.

Proof: At the end Step 2, $\bar{\lambda}_1(r) = \frac{\lambda_0 b_l}{\varepsilon_1} r^{\frac{l}{\varepsilon_1}-1},\ r \geq 0$, and $\bar{\lambda}_2(r)$ satisfies the equation $\bar{\lambda}_2(r)\beta(r) = \bar{\lambda}_1(r)$, where $\beta(r) = \begin{cases} 1 & ,\ r \leq 1 \\ \frac{\varepsilon_1}{\varepsilon_2} r^{\frac{l}{\varepsilon_2}-\frac{l}{\varepsilon_1}} & ,\ r > 1 \end{cases}$. Since $\varepsilon_2 > \varepsilon_1 > l$, $\beta(r)$ is a non increasing function. As a result, Corollary 4 holds and hence $\frac{C}{I}\Big|_{\bar{\lambda}_1(r)} \leq_{\text{st}} \frac{C}{I}\Big|_{\bar{\lambda}_2(r)}$. Thus, the system with the path-loss model $\frac{1}{h_2(r)}$ has a better signal quality compared to that of $\frac{1}{h_1(r)}$. Now, when $\varepsilon_1 > \varepsilon_2 > l$, $\beta(r)$ is a non decreasing function and $\frac{C}{I}\Big|_{\bar{\lambda}_1(r)} \geq_{\text{st}} \frac{C}{I}\Big|_{\bar{\lambda}_2(r)}$. ■

D. Shadow Fading and Random Transmission Powers

In all the results until now, the shadow fading factors and the transmission powers for all the BSs were constant. Here, we generalize to the case when they are random variables, i.i.d. across BSs. The transmission power and the shadow fading factor of the same BS could be dependent. The following result reduces the 1-D SCS with random shadow fading factors and random transmission powers to a 1-D SCS where both are 1.

Theorem 2. For the $\frac{C}{I}$ analysis in a 1-D SCS with BS density function $\lambda(r)$, if the random shadow fading factors $\{\Psi_i\}_{i=1}^{\infty}$ and transmission powers $\{K_i\}_{i=1}^{\infty}$ are i.i.d. across all the BSs, the SCS is equivalent to another 1-D SCS with BS density function $\bar{\lambda}(r) = \mathbb{E}_{\Psi,K}\left[(\Psi K)^{\frac{1}{\varepsilon}}\lambda\left(r(\Psi K)^{\frac{1}{\varepsilon}}\right)\right]$, where $\mathbb{E}$ is the expectation operator w.r.t. Ψ and K, which has the same distribution as Ψ_i and $K_i,\ \forall\ i$, respectively. This holds as long as the expectation converges.

Proof: For the random shadow fading and transmission powers case, the $\frac{C}{I}$ defined in Section II can be written as $\frac{C}{I} = \frac{\left(R_j(K_j\Psi_j)^{-\frac{1}{\varepsilon}}\right)^{-\varepsilon}}{\sum\limits_{i,\ i\neq j}\left(R_i(K_i\Psi_i)^{-\frac{1}{\varepsilon}}\right)^{-\varepsilon}}$, where the index j corresponds to the BS

with the strongest received signal power at the MS. The above expression for the $\frac{C}{I}$ corresponds to a 1-D SCS with distances from the MS given by $\left\{R_i\left(K_i\Psi_i\right)^{-\frac{1}{\varepsilon}}\right\}_{i=1}^{\infty}$, unity shadow fading factors and unity transmission powers at each BS. Now, [2, Theorem 4] applies, and the new 1-D SCS has a BS density function $\bar{\lambda}\left(r\right)$. ■

Thus, if there are random shadow fading factors and/or transmission powers at each BS, one can first apply Theorem 2 to obtain the equivalent 1-D SCSs with constant shadowing and transmission powers, and then apply Theorem 1 for the comparison of $\frac{C}{I}$'s. The following corollary shows a scenario where $\frac{C}{I}$ distribution is unaffected by the distributions of random shadow fading factors and transmission powers.

Corollary 7. In a *homogeneous* $l-$D SCS with a BS density λ_0, the $\frac{C}{I}$ distribution at the MS does not depend on the distributions of the random shadow fading factors $\{\Psi_i\}_{i=1}^{\infty}$ and transmission powers $\{K_i\}_{i=1}^{\infty}$, if they are i.i.d. across BSs and $\left|\mathbb{E}_{\Psi,K}\left[\left(\Psi K\right)^{\frac{l}{\varepsilon}}\right]\right| < \infty$.

Proof: Firstly, recall that the homogeneous $l-$D SCS is equivalent to a 1-D SCS with a BS density function $\lambda\left(r\right)=\lambda_0 b_l r^{l-1},\ r\geq 0$. We have $\left(\frac{C}{I}\right)_{\text{rand}} \overset{(a)}{=}_{\text{st}} \frac{C}{I}\Big|_{\bar{\lambda}(r)} \overset{(b)}{=}_{\text{st}} \frac{C}{I}\Big|_{\frac{1}{\alpha}\lambda\left(\frac{r}{\alpha}\right)} \overset{(c)}{=}_{\text{st}} \frac{C}{I}\Big|_{\lambda(r)}$, where (a) is obtained by applying Theorem 2 to the 1-D SCS with the BS density function $\lambda\left(r\right)$, to obtain the equivalent 1-D SCS with constant shadow fading factors and transmission powers and with a BS density $\bar{\lambda}\left(r\right)=\mathbb{E}_{\Psi,K}\left[\left(\Psi K\right)^{\frac{1}{\varepsilon}}\lambda\left(r\left(\Psi K\right)^{\frac{1}{\varepsilon}}\right)\right]=\mathbb{E}_{\Psi,K}\left[\left(\Psi K\right)^{\frac{l}{\varepsilon}}\right]\lambda\left(r\right)$; (b) is obtained by rewriting $\bar{\lambda}\left(r\right)$ as $\frac{1}{\alpha}\lambda\left(\frac{r}{\alpha}\right)$ where $\alpha=\left(\mathbb{E}_{\Psi,K}\left[\left(\Psi K\right)^{\frac{l}{\varepsilon}}\right]\right)^{-\frac{1}{l}}$; (c) is obtained by applying Corollary 1. Thus, the shadow fading and random transmission powers have no effect on the $\frac{C}{I}$ distribution. ■

This result was already proved in [2, Remark 4(a)]. But here, we have shown an elegant alternative proof that is based only on the concepts of stochastic ordering. Finally, as a consequence of Corollary 7, the results in Section IV-C also hold for cases with random shadow fading factors and transmission powers that are i.i.d. across BSs.

V. Conclusions

This paper is an extension to our previous work in characterizing the $\frac{C}{I}$ of a SCS in [2]. The study of the $\frac{C}{I}$ at the MS in a cellular system with BSs distributed according to a *non-homogeneous* Poisson process is difficult because the distribution of the $\frac{C}{I}$ is not in closed form [2] and it is difficult to form an intuition about such networks. As a result, most of the $\frac{C}{I}$ analysis

are restricted to the *homogeneous* $l-D$ SCSs. Here, we have developed a stochastic ordering based tool to analyze the $\frac{C}{I}$ at the MS in such non-homogeneous Poisson processes. Due to Theorem 1, for certain BS densities, we show that, by just comparing the BS density functions of the SCSs, we can make strong inferences such as, a MS in a given SCS achieves a $\frac{C}{I}$ that is at least as good as that achieved in another SCS without having to solve for the $\frac{C}{I}$ distributions. Moreover, as a consequence of Theorem 1, elegant proofs are derived to show that (1) a MS in a *homogeneous* l-D SCS sees decreasing signal quality as dimension l increases; (2) the $\frac{C}{I}$ at the MS improves as the path-loss exponent of the channel increases; and (3) as far as $\frac{C}{I}$ is concerned, a SCS with random shadow fading factors and random transmission powers, which are i.i.d. across BSs, is equivalent to a SCS with constant shadow fading factors and transmission powers and with a modified BS density.

APPENDIX

A. Proof of Theorem 1

Consider the 1-D SCS specified by the set $\{\lambda(r), \mu(r), \mu^{-1}(q)\}$, as in Definition 2. The following remark relates $\frac{C}{I}$ to the cumulative BS density.

If R_1 denotes the distance between the serving BS and MS in the 1-D SCS,

$$\mathbb{P}\left(\left\{\frac{C}{I}>y\right\}\right) \overset{(a)}{=} \int_{r_1=0}^{\infty} \mathbb{P}\left(\frac{C}{I}>y \middle| R_1=r\right) f_{R_1}(r)dr \overset{(b)}{=} \int_{q=0}^{\infty} \mathbb{P}\left(\frac{C}{I}>y \middle| Q=q\right) f_Q(q)dq,$$

where $Q \triangleq \mu(R_1)$, and Q is an exponential random variable with mean 1.

Equation (a) is obtained by conditioning w.r.t. R_1. Equation (b) is obtained by expressing (a) in terms of Q, where the p.d.f. of R_1 at $R_1 = \mu^{-1}(q)$ is $f_{R_1}(r)dr|_{r=\mu^{-1}(q)} = e^{-\int_0^r \lambda(s)ds}\lambda(r)dr\big|_{r=\mu^{-1}(q)} = e^{-q}dq = f_Q(q)dq$, which does not depend on $\lambda(r)$.

To show that the BS density $\lambda_1(r)$ gives a worse $\frac{C}{I}$ than $\lambda_2(r)$ does, one needs to show that $\frac{C}{I}\Big|_{R_1=\mu_1^{-1}(q),\ \lambda_1(r),\ r\geq\mu_1^{-1}(q)} \leq_{\text{st}} \frac{C}{I}\Big|_{R_1=\mu_2^{-1}(q),\ \lambda_2(r),\ r\geq\mu_2^{-1}(q)}$ for all $q>0$, where the condition of the domain of the BS density is because the locations of interfering BSs only depend on the BS density in that domain. Next, define $a=\frac{\mu_2^{-1}(q)}{\mu_1^{-1}(q)}$. By Corollary 1,

$$\begin{aligned}\frac{C}{I}\Bigg|_{R_1=\mu_1^{-1}(q),\ \lambda_1(r),\ r\geq\mu_1^{-1}(q)} &= \frac{\left(\mu_1^{-1}(q)\right)^{-\varepsilon}}{\sum_{k=2}^{\infty} R_k^{-\varepsilon}}\Bigg|_{R_2\geq\mu_1^{-1}(q),\ \lambda_1(r),\ r\geq\mu_1^{-1}(q)} \\ = \frac{\left(a\mu_1^{-1}(q)\right)^{-\varepsilon}}{\sum_{k=2}^{\infty}(aR_k)^{-\varepsilon}}\Bigg|_{R_2\geq\mu_1^{-1}(q),\ \lambda_1(r),\ r\geq\mu_1^{-1}(q)} &=_{\text{st}} \frac{\left(\mu_2^{-1}(q)\right)^{-\varepsilon}}{\sum_{k=2}^{\infty}\left(R_k'\right)^{-\varepsilon}}\Bigg|_{R_2'\geq\mu_2^{-1}(q),\ \frac{1}{a}\lambda_1(\frac{r}{a}),\ r\geq\mu_2^{-1}(q)},\end{aligned}$$

where R_k''s are the ordered BS locations of the SCS with BS density $\frac{1}{a}\lambda\left(\frac{r}{a}\right)$. The equation means that the conditional $\frac{C}{I}$ of the SCS with a BS density $\lambda_1(r)$ is equivalent to an SCS with BS density $\frac{1}{a}\lambda_1(\frac{r}{a})$ with the same location of the serving BS as the SCS with BS density $\lambda_2(r)$.

With the locations of the serving BSs equal and fixed, $\frac{C}{I}$ is a decreasing function of the interference. Theorem 1.A.3.(a) of [4] says that decreasing functions reverse stochastic order. Therefore, one only needs to show that the interferences satisfy

$$\sum_{k=2}^{\infty} R_k^{-\varepsilon}|_{R_2 \geq \mu_2^{-1}(q),\ \frac{1}{a}\lambda_1(\frac{r}{a}),\ r \geq \mu_2^{-1}(q)} \quad \geq_{\text{st}} \quad \sum_{k=2}^{\infty} R_k^{-\varepsilon}|_{R_2 \geq \mu_2^{-1}(q),\ \lambda_2(r),\ r \geq \mu_2^{-1}(q)}. \tag{3}$$

As shown in [2, Appendix B], the total interference power can be expressed as $\sum_{k=2}^{\infty} R_k^{-\varepsilon} = \lim_{r_B \to \infty} \lim_{N \to \infty} \sum_{i=2}^{N} X_i$, where X_i is a Bernoulli random variable defined by

$$\mathbb{P}\left(\left\{ X_i = 0 \right| R_1 = r_1 \right\}) = p_i,\ \mathbb{P}\left(\left\{ X_i = r_i^{-\varepsilon} + o\left(\Delta r\right) \right| R_1 = r_1 \right\}\right) = 1 - p_i,$$

$p_i = \lambda(r_i)\Delta r + o(\Delta r)$, $r_i = r_1 + (i-1)\Delta r$, $\Delta r = \frac{r_B - r_1}{N}$, and $r_1 = \mu_2^{-1}(q)$. Now, since the condition $\frac{1}{a}\lambda_1(\frac{r}{a}) \geq \lambda_2(r)$ holds for all $r \geq \mu_2^{-1}(q)$, we have $X_i|_{\frac{1}{a}\lambda_1(\frac{r}{a})} \geq_{\text{st}} X_i|_{\lambda_2(r)},\ \forall i \geq 2$. Since summation preserves the stochastic order [4, Theorem 1.A.3.(b)], (3) is proved, completing the proof of Theorem 1.

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